Probability Distributions of the Relative Saturation and Saturated Areas of a River Basin

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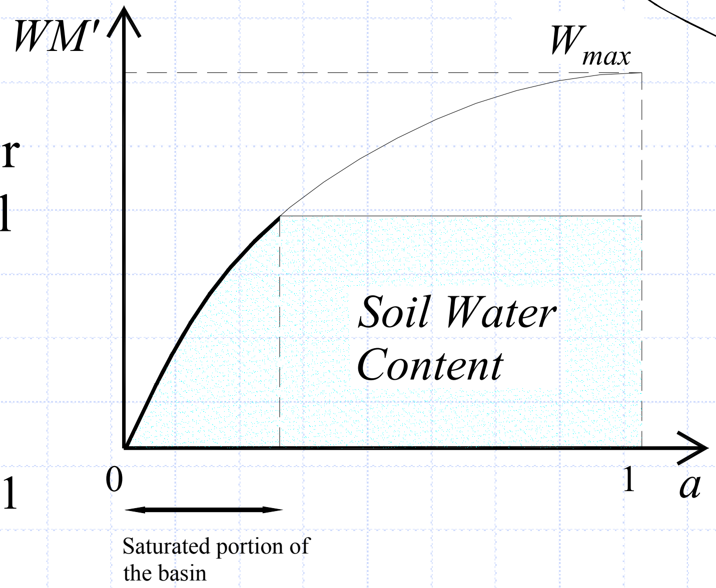
Water Balance Dynamics in a River Basin: Outlines

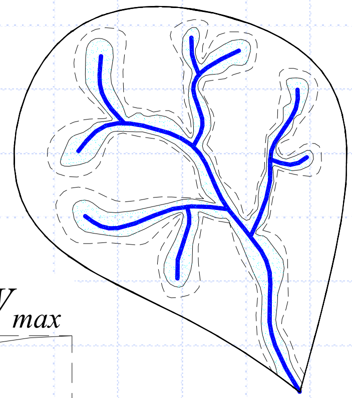
- Soil moisture dynamics driven by a stochastic rainfall forcing;
- Dynamics of soil water balance in a schematic river basin;
- Effects due to the soil heterogeneity in basin response;
- Investigation on the dynamics of significant hydrological variable such as the saturated portion of a basin and of its relative saturation;
- Investigation on the dynamics of probability distribution of the produced runoff.

Model Assumptions

The watershed heterogeneity is described using a parabolic curve for the water storage capacity of the soil (*Zhao et al.*, 1980)

$$\frac{f}{F} = 1 - \left(1 - \frac{WM'}{W_{max}}\right)^{b} \qquad \text{Eq.1}$$





where *f/F* represents the fraction of the basin with water storage capacity ≤*WM'*, *WMM* represents the maximum value of the water storage capacity in the basin and *b* is a shape parameter that according to *Zhao* (1992) assumes values between 0.1-0.4 increasing with the characteristic dimension of the basin.

$$WM = \frac{W_{max}}{1+b}$$ is the total water storage capacity.

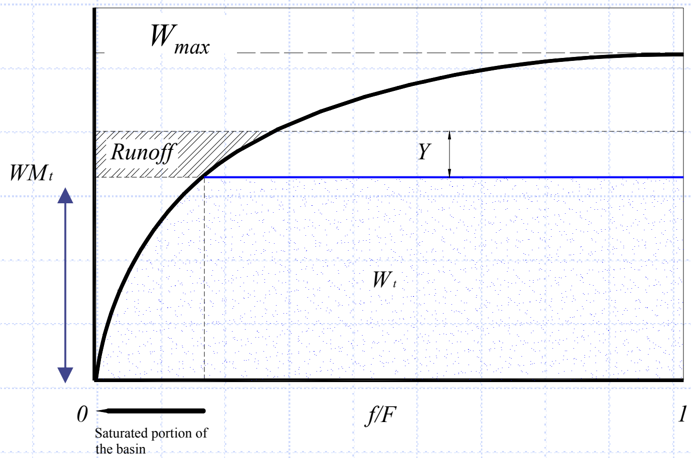
Definitions

The <u>watershed-average soil moisture storage</u> at time *t*, is the integral of 1-*f/F* between zero and WM^*_t water content

$$W_t = \int_0^{WM_t^*} \left(1 - \frac{f}{F}\right) dWM' = WM\left(1 - \left(1 - \frac{WM_t^*}{W_{max}}\right)^{1+b}\right) \qquad \text{Eq.2}$$

A relevant variable for this problem is given by the relative saturation, *s*, expressed as the ratio between watershed-average soil moisture storage and the total available volume

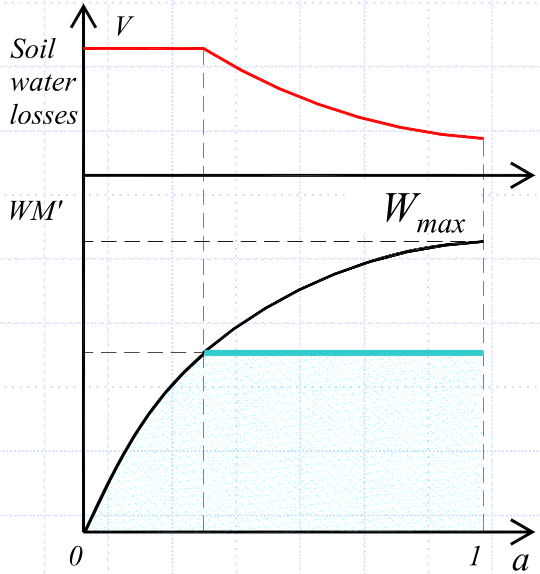
$$s = \frac{W_t}{WM} = \left(1 - \left(1 - \frac{WM_t^*}{W_{max}}\right)^{1+b}\right) \qquad \text{Eq.3}$$



The relative water level

$$R = \frac{WMt}{W_{max}\, t} \qquad \text{Eq.4}$$

Soil Water Losses

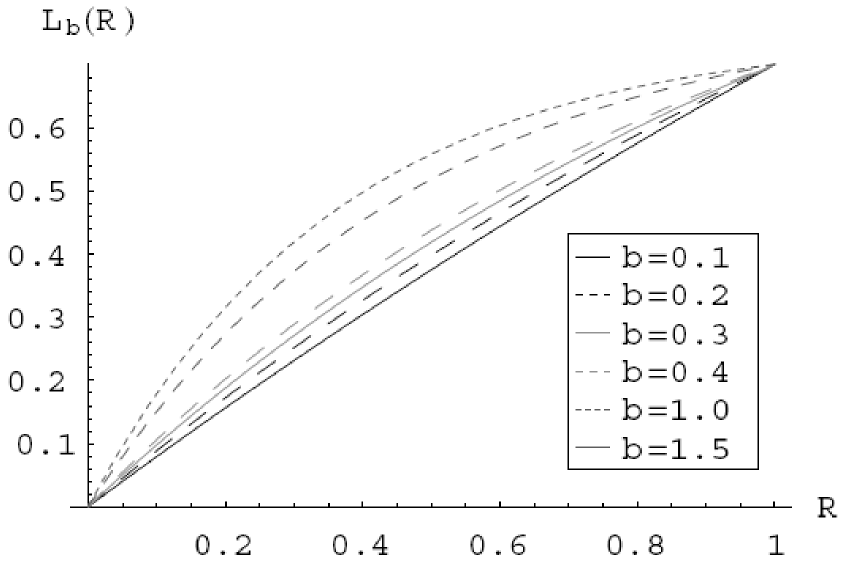


A possible approximation for the sum of soil leakage and actual evapotranspiration is given by a linear function (e.g., *Pan et al.*, 2003; *Porporato et al.*, 2004) where the soil losses are assumed to be proportional to the soil water content in a point

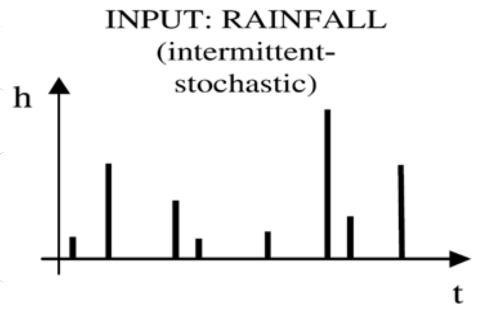
$$L(s,x) = V\zeta(t,x) \qquad \text{Eq.5}$$

that, at the basin scale, becomes

$$L_b(WM_t) = Vs = V\left(1-\left(1-\frac{WM_t}{W_{max}'}\right)^{1+b}\right)$$



$$L_b\left(R=\frac{WM_t}{W_{max}'}\right) \cong V\left(\frac{e^{-kR}-1}{e^{-k}-1}\right) \qquad \text{Eq.7}$$



Soil Water Balance at Basin Scale

The soil water balance over the basin can be described through the following differential equation in WM_t

$$\frac{dWM_t}{dt} = Y - Vs = Y - V\left(\frac{e^{-k\frac{WM_t}{W_{max}}} - 1}{e^{-k} - 1}\right) \qquad \text{Eq.8}$$

where Y represents an additive term of infiltration (poisson process) and water losses are assumed to be proportional to the relative saturation of the basin s.

The above differential equation after the standardization of the variables becomes

$$W_{max}\frac{dR}{\mathrm{dt}} = Y - V(1 - (1 - R)^{1+b}) \qquad \text{Eq.9}$$

with $R = \frac{WMt}{W_{max}}$

Solution at the Steady State

Following Rodrìguez-Iturbe et al. (1999), the steady state probability density function of R with the simplified loss function $\rho(R)$ can be obtained as

$$p(R) = \frac{C}{\rho(R)} e^{-\gamma R + \lambda \int \frac{1}{\rho(R)} du} = \frac{C e^{k(R-1) - R\gamma} \left(e^k - 1\right) \left(e^{kR} - 1\right)^{\frac{\lambda(1-e^{-k})}{k\beta} - 1}}{\beta} \quad \text{Eq.10}$$

the constant C_1 assumes the following value

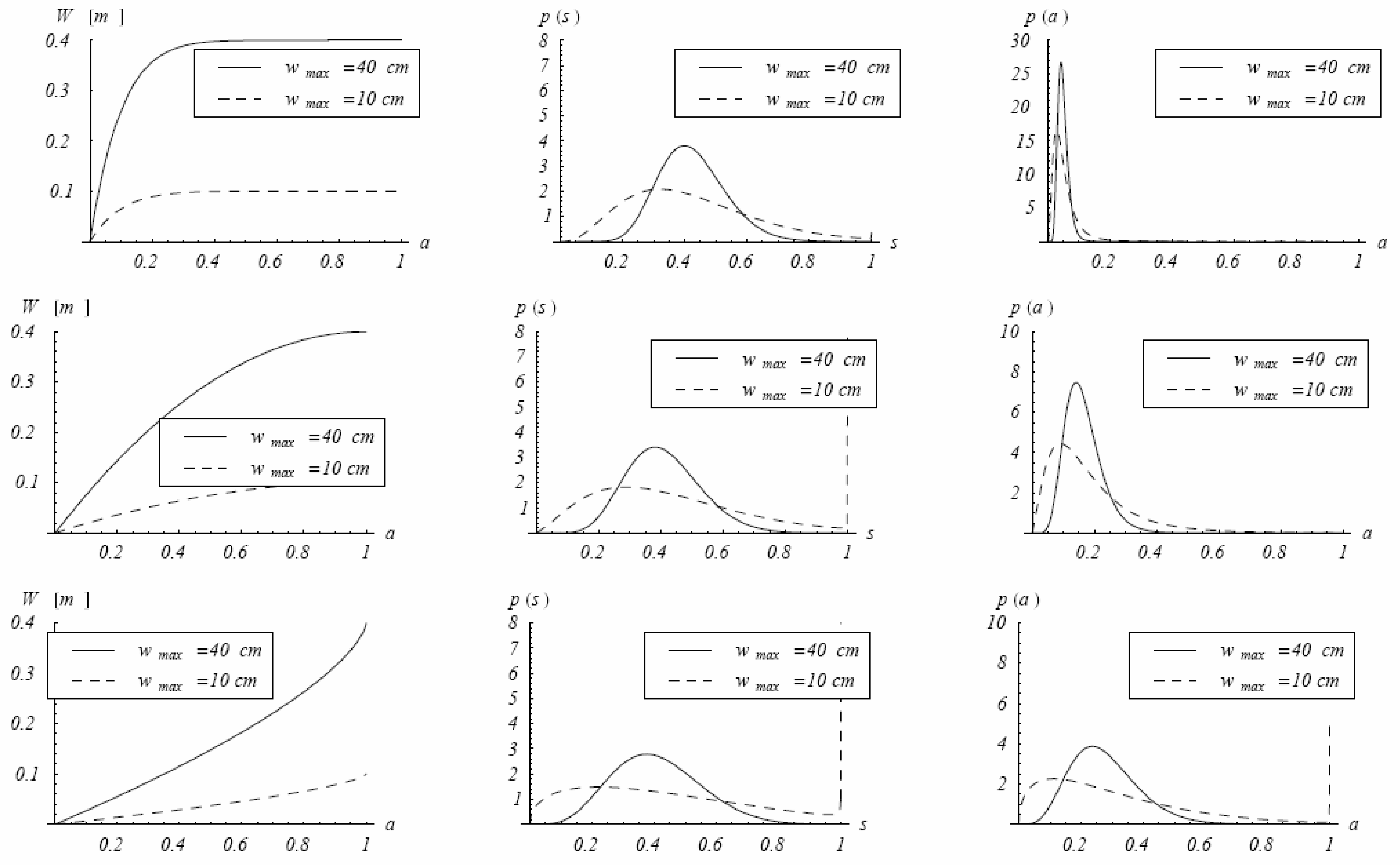
$$C_1 = 1 / \int_0^1 \frac{e^{k(-1+R) - R\gamma} \left(e^k - 1\right) \left(e^{kR} - 1\right)^{-1 + \frac{\lambda - e^{-k}\lambda}{k\beta}}}{\beta} dR \quad \text{Eq.11}$$

Under these hypotheses, it is possible to define the probability distribution of saturated areas from the probability density function of R as

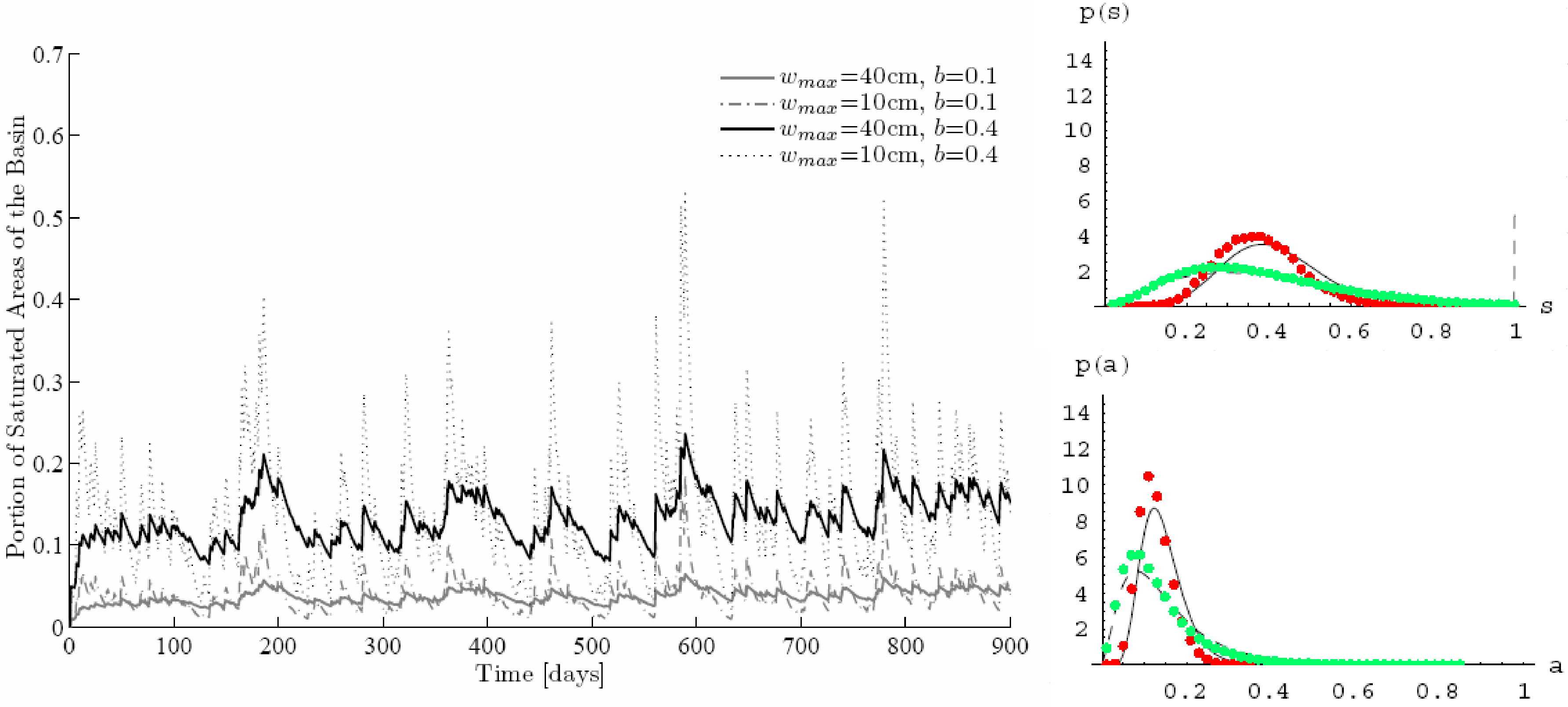
$$p_A(a) = p_R(f^{-1}(a)) \frac{df^{-1}(a)}{da} \quad \text{Eq.12}$$

where $R = f^{-1}(a) = 1 - (1 - a)^{\frac{1}{b}}$

Results: PDFs of *s* and *a*



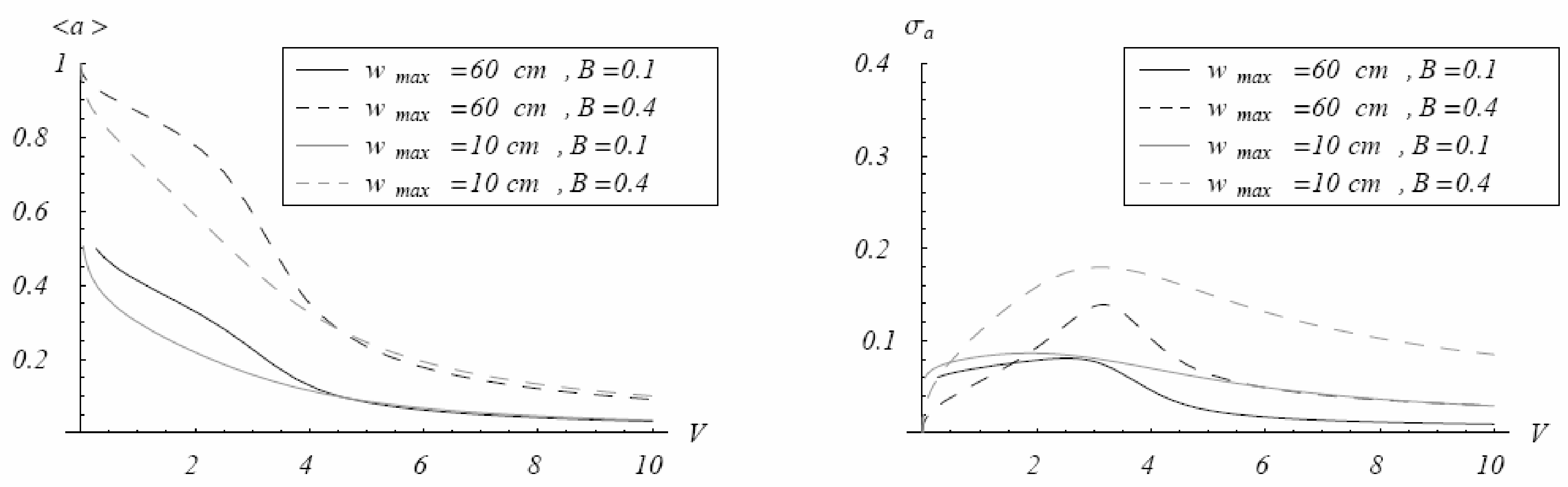
Numerical Simulation



Temporal dynamics of saturated areas for different values of w_{max} and b reproduced by a numerical simulation performed at the daily time-scale.

Expected Value and Standard Deviation of *a*

Expected value and standard deviation of the saturated area as a function of the soil water losses coefficient V



Saturated areas reach the maximum variance when the soil water losses coefficient is equal to the mean daily rainfall.

Probability Distribution of Runoff

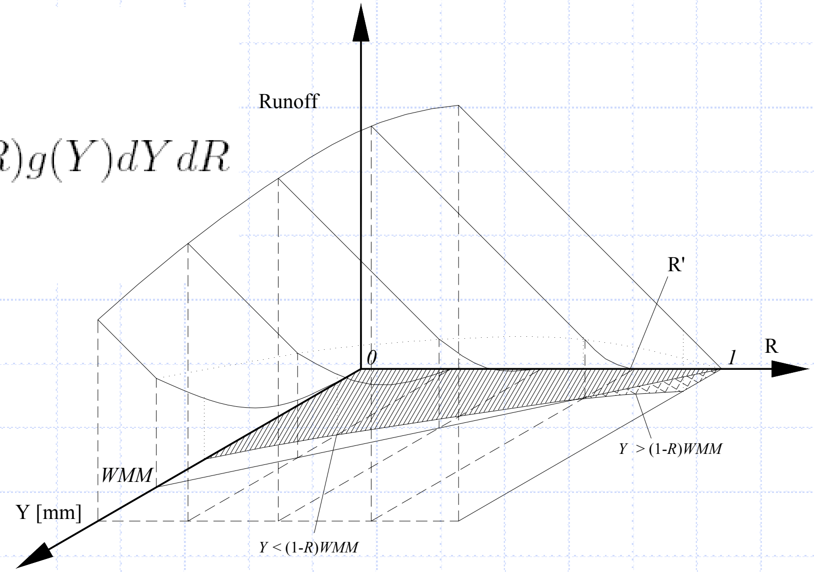
In the present scheme, the runoff can be described through the following equation

$$
\begin{array}{ll}
q = Y - \frac{WMM}{1+b}\left((1-R)^{1+b} - (1 - R - \frac{Y}{WMM})^{1+b}\right) & Y \leq (1-R)WMM \\
q = Y - \frac{WMM}{1+b}(1-R)^{1+b} & Y \geq (1-R)WMM
\end{array}
\qquad \text{Eq.13}
$$

In order to derive the probability distribution of runoff, one should integrate the join probability distribution of rainfall, *Y*, and *R*

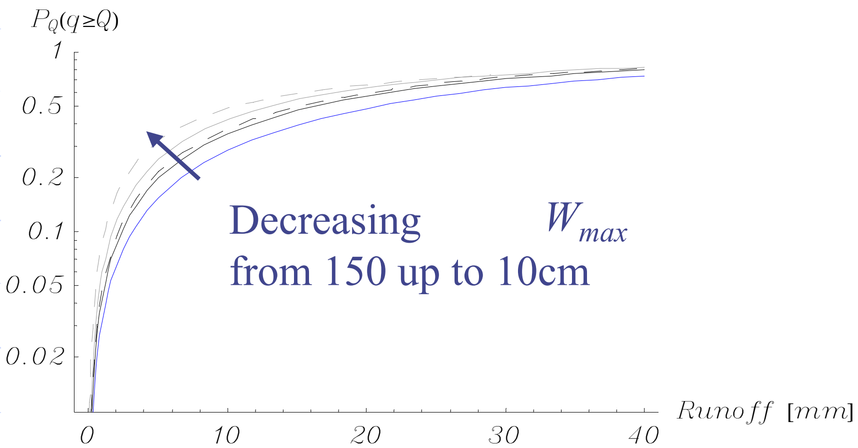
$$
P_Q(q) = \int_0^{R_1} \int_0^{\frac{\mathrm{WMM}\left(\left((1-R)^b-1\right)(1-R)+\sqrt{\frac{2bq(1-R)^{1+b}}{\mathrm{WMM}}+\left((1-R)^b-1\right)^2(R-1)^2}\right)}{b(1-R)^b}} g(R)g(Y)dY\,dR + \int_{R_1}^{1} \int_0^{q+\frac{\mathrm{WMM}(1-R)^{1+b}}{1+b}} g(R)g(Y)dY\,dR
$$

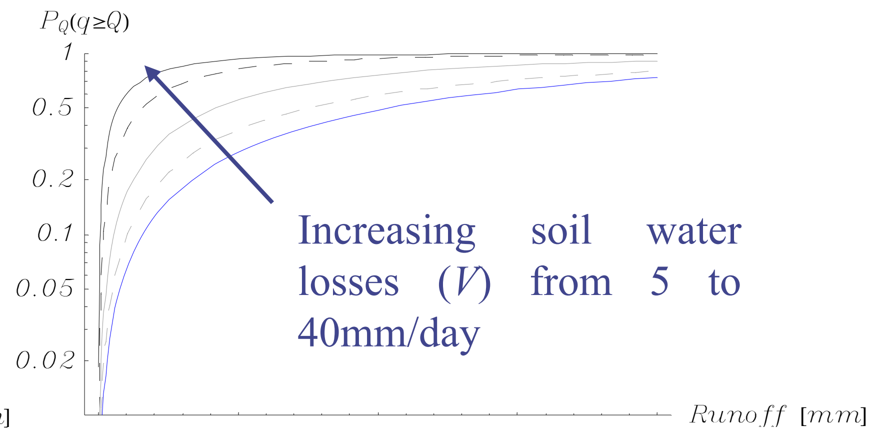
$$
R_1 \simeq \frac{1-b}{6} + \frac{1}{6}\sqrt{(25+10b+b^2) - \frac{q}{\mathrm{WMM}}(24\frac{1}{b} + 36 + 12b)}
$$

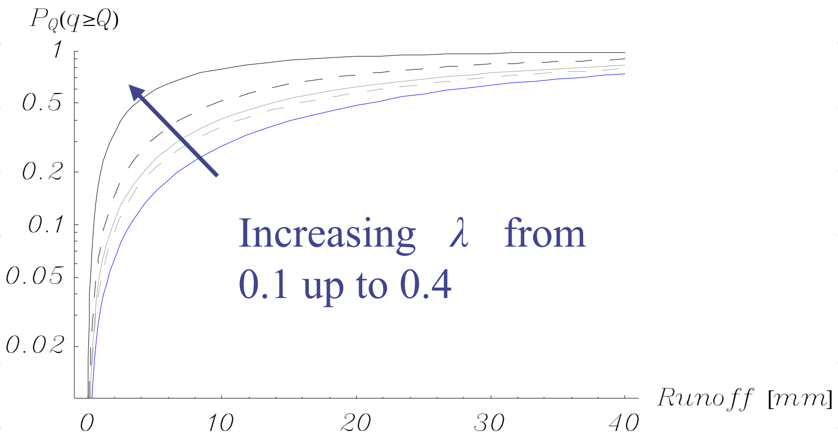


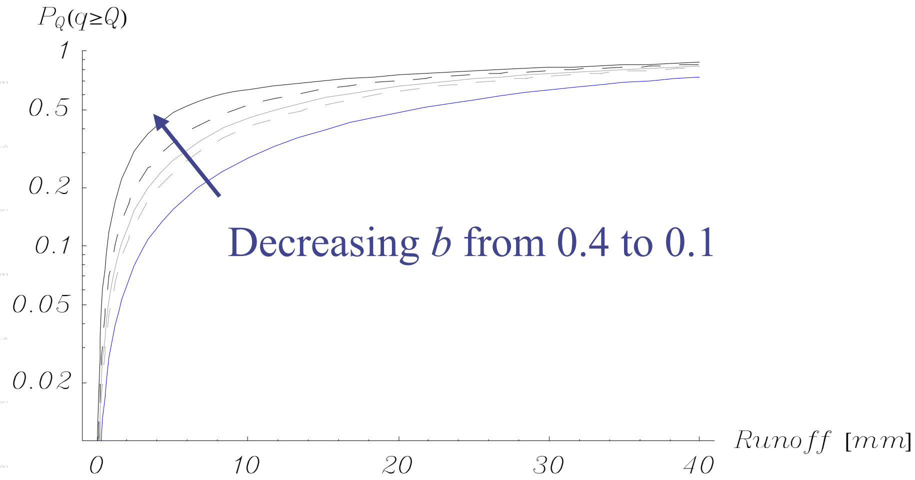
Runoff Probability Distributions

— Rainfall CDF









Conclusion

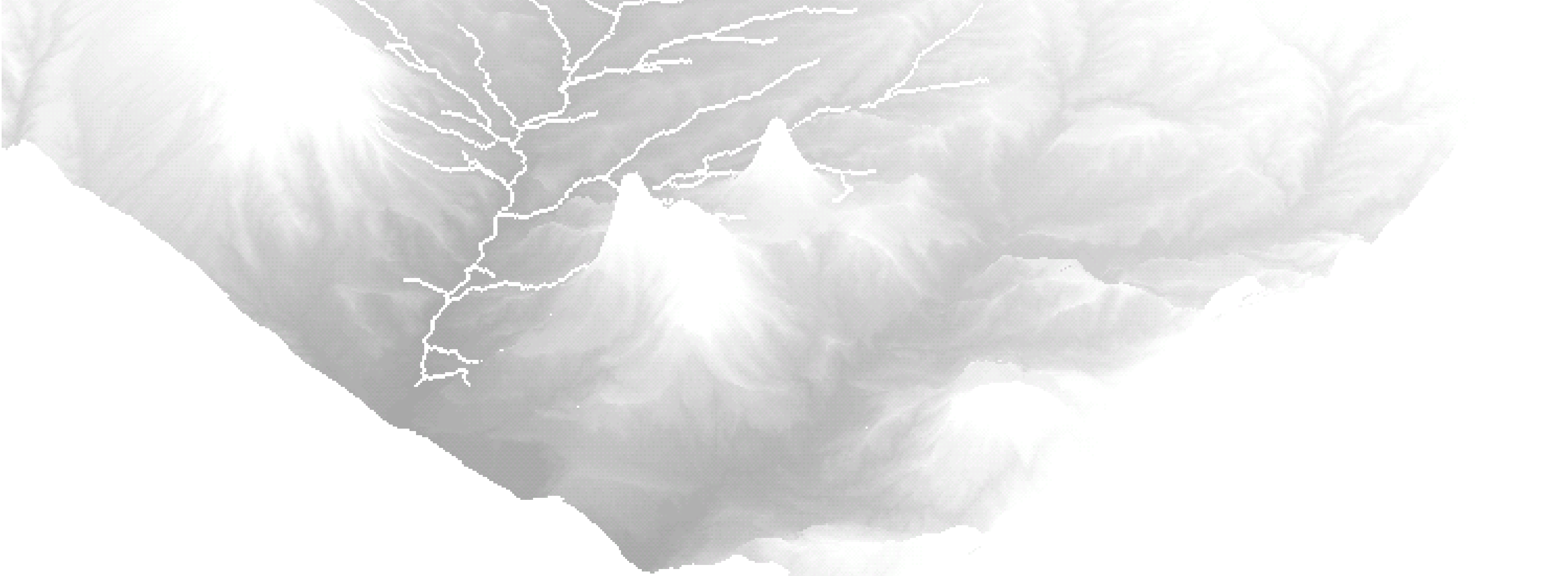
- Definition of a feasible mathematical characterization of soil moisture dynamics at the basin scale (humid environment);
- Effects due to the soil heterogeneity in basin response;
- Derivation of the probability density function of the saturated portion of a basin and of its relative saturation;
- Derivation of the cumulative probability distribution of the produced runoff with a clear dependence from climatic and physiographic basin characteristics.

Papers related to this research line…

Manfreda, S., M. Fiorentino, A Stochastic Approach for the Description of the Water Balance Dynamics in a River Basin, *Hydrol. Earth Syst. Sci. Discuss.*, 5, 723-748, 2008.

Manfreda, S., M. McCabe, E.F. Wood, M. Fiorentino and I. Rodríguez-Iturbe, Spatial Patterns of Soil Moisture from Distributed Modeling, *Advances in Water Resources*, 30(10), 2145-2150, 2007, (doi: 10.1016/j.advwatres.2006.07.009).

Fiorentino, M., S. Manfreda, V. Iacobellis, Peak Runoff Contributing Area as Hydrological Signature of the Probability Distribution of Floods, *Advances in Water Resources*, 30(10), 2123-2144, 2007 (doi: 10.1016/ j.advwatres.2006.11.017).



Thanks for your attention…